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# ON THE SIGNIFICANCE OF THE WEYL CURVATURE IN A RELATIVISTIC COSMOLOGICAL MODEL

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The Weyl curvature includes the Newtonian field and an additional field, the so-called anti-Newtonian. In this paper, we use the Bianchi and Ricci identities to provide a set of constraints and propagations for the Weyl fields. The temporal evolutions of propagations manifest explicit solutions of gravitational waves. We see that models with purely Newtonian field are inconsistent with relativistic models and obstruct sounding solutions. Therefore, both fields are necessary for the nonlocal nature and radiative solutions of gravitation.

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## 1. Introduction

In the theory of general relativity, one can split the Riemann curvature tensor into the Ricci tensor defined by the Einstein equation and the Weyl curvature tensor.[1–4] Additionally, one can split the Weyl tensor into the electric part and the magnetic part, the so-called gravitoelectric/magnetic fields,[5] being due to some similarity to electrodynamical counterparts.[2,6–9] We describe the gravitoelectric field as the tidal (Newtonian) force,[9,10] but the gravitomagnetic field has no Newtonian analogy, called anti-Newtonian. Nonlocal characteristics arising from the Weyl curvature provides a description of the Newtonian force, although the Einstein equation describes a local dynamics of spacetime.[9,11] The Weyl curvature also includes an additional force: the gravitomagnetic field that is produced by the mass currents analogously an electric current generating a magnetic field.[2] In fact, the theory of general relativity predicts two main concepts: gravitomagnetic fields and gravitational waves. Gravitation similar to electromagnetism propagates at identical speed,

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that provides a sounding analysis and a radiative description of force. We notice the Weyl tensor encoding the tidal force, a new force by its magnetic part, and a treatment of gravitational waves.

Determination of gravitational waves and gravitomagnetism (new force) is experimental tests of general relativity.[12] Gravitational radiation of a binary system of compact objects has been proposed to be detected by a resonant bar[13] or a laser interferometer in space,[14,15] such as the LIGO[16] and VIRGO.[17] A non-rotating compact object produces the standard Schwarzschild field, whereas a rotating body also generates the gravitomagnetic field. It has been suggested as a mechanism for the jet formation in quasars and galactic nuclei.[18,19] The resulting action of the gravitomagnetic fields and of the viscous forces implies that the formation of the accretion disk into the equatorial plane of the central body while the jets are ejected along angular momentum vector perpendicularly to the equatorial plane.[2,18] The gravitomagnetic field implies that a rotating body e. g. the Earth affects the motion of orbiting satellites. This effect has been recently measured using the LAGEOS I and LAGEOS II satellites.[20] However, we may need to count some possible errors in the LAGEOS data.[21] Using two recent orbiting geodesy satellites (CHAMP and GRACE), it has been reported confirmation of general relativity with a total error between 5% and 10%.[22–24]

In this paper, we describe kinematic and dynamic equations of the Weyl curvature variables, i.e. the gravitoelectric field as the relativistic generalization of the tidal forces and the gravitomagnetic field in a cosmological model containing the relativistic fluid description of matter. We use the convention based on $8\pi G = 1 = c$. We denote the round brackets enclosing indices for symmetrization, and the square brackets for antisymmetrization. The organization of this paper is as follows. In Sec. 2, we introduce the $3 + 1$ covariant formalism, kinematic quantities, and dynamic quantities in a hydrodynamic description of matter. In Sec. 3, we obtain constraint and propagation equations for the Weyl fields from the Bianchi and Ricci identities. In Sec. 4, rotation and distortion are characterized as wave solutions. In Sec. 5, we study a Newtonian model as purely gravitoelectric in an irrotational static spacetime and a perfect-fluid model, and an anti-Newtonian model as purely gravitomagnetic in a shearless static and perfect-fluid model. We see that both models are generally inconsistent with relativistic models, allowing no possibility for wave solutions. Section 6 provides a conclusion.

## 2. Covariant Formalism

According to the pattern of classical hydrodynamics, we decompose the spacetime metric into the spatial metric and the instantaneous rest-space of a comoving observer. The formalism, known as the $3 + 1$ covariant approach to general relativity,[25–30] has been used for numerous applications.[10,31–35] In this approach, we rewrite equations governing relativistic fluid dynamics by using projected vectors and projected symmetric traceless tensors instead of metrics.[10,34]

We take a four-velocity vector $u^a$ field in a given $(3+1)$-dimensional spacetime to be a unit vector field $u^a u_a = -1$. We define a spatial metric (or projector tensor) $h_{ab} = g_{ab} + u_a u_b$, where $g_{ab}$ is the spacetime metric. It decomposes the spacetime metric into the spatial metric and the instantaneous rest-space of an observer moving with four-velocity $u^a$.[2,33,36] We get some properties for the spatial metric

$$h_{ab}u^b = 0\,, \qquad h_a{}^c h_{cb} = h_{ab}\,, \qquad h_a{}^a = 3\,. \tag{1}$$

We also define the spatial alternating tensor as

$$\varepsilon_{abc} = \eta_{abcd} u^d\,, \tag{2}$$

where $\eta_{abcd}$ is the spacetime alternating tensor,

$$\eta_{abcd} = -4!\sqrt{|g|}\delta^0{}_{[a}\delta^1{}_b\delta^2{}_c\delta^3{}_{d]}\,, \qquad \delta_a{}^b = g_{ac}g^{cb}\,, \qquad |g| = \det g_{ab}\,. \tag{3}$$

The covariant spacetime derivative $\nabla_a$ is split into a covariant temporal derivative

$$\dot{T}_{a\ldots} = u^b \nabla_b T_{a\ldots} \tag{4}$$

and a covariant spatial derivative

$$\mathrm{D}_b T_{a\ldots} = h_b{}^d h_a{}^c \cdots \nabla_d T_{c\ldots}\,. \tag{5}$$

The projected vectors and the projected symmetric traceless parts of rank-2 tensors are defined by

$$V_{\langle a\rangle} \equiv h_a{}^b V_b\,, \qquad S_{\langle ab\rangle} \equiv \left\{ h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h^{cd} h_{ab} \right\} S_{cd}\,. \tag{6}$$

The equations governing these quantities involve a vector product and its generalization to rank-2 tensors:

$$[V, W]_a \equiv \varepsilon_{abc} V^b W^c\,, \qquad [S, Q]_a \equiv \varepsilon_{abc} S^b{}_d Q^{cd}\,, \tag{7}$$

$$[V, S]_{ab} \equiv \varepsilon_{cd(a} S_{b)}{}^c V^d\,, \qquad [V, S]_{\langle ab\rangle} \equiv \varepsilon_{cd\langle a} S_{b\rangle}{}^c V^d\,. \tag{8}$$

We define divergences and rotations as

$$\mathrm{div}(V) \equiv \mathrm{D}^a V_a\,, \qquad (\mathrm{div}\, S)_a \equiv \mathrm{D}^b S_{ab}\,, \tag{9}$$

$$\begin{aligned} &(\mathrm{curl}\, V)_a \equiv \varepsilon_{abc}\mathrm{D}^b V^c\,, \qquad (\mathrm{curl}\, S)_{ab} \equiv \varepsilon_{cd(a}\,\mathrm{D}^c S_{b)}{}^d\,, \\ &(\mathrm{curl}\, S)_{\langle ab\rangle} \equiv \varepsilon_{cd\langle a}\,\mathrm{D}^c S_{b\rangle}{}^d\,. \end{aligned} \tag{10}$$

We know that $\mathrm{D}_c h_{ab} = 0 = \mathrm{D}_d \varepsilon_{abc}$, $\dot{h}_{ab} = 2u_{(a}\dot{u}_{b)}$ and $\dot{\varepsilon}_{abc} = 3u_{[a}\varepsilon_{bc]d}\dot{u}^d$, then $u^a \dot{h}_{ab} = -\dot{u}_b$ and $u^a \dot{\varepsilon}_{abc} = -\dot{u}^a \varepsilon_{abc}$. From these points one can also define the relativistically temporal rotations as

$$\begin{aligned} &[\dot{u}, V]_a = -u^c \dot{\varepsilon}_{abc} V^b\,, \qquad [\dot{u}, S]_{ab} = -u^c \dot{\varepsilon}_{cd(a} S_{b)}{}^d\,, \\ &[\dot{u}, S]_{\langle ab\rangle} = -u^c \dot{\varepsilon}_{cd\langle a} S_{b\rangle}{}^d\,. \end{aligned} \tag{11}$$

The covariant spatial distortions are

$$\mathrm{D}_{\langle a}V_{b\rangle} = \mathrm{D}_{(a}V_{b)} - \frac{1}{3}(\operatorname{div} V)h_{ab}\,, \tag{12}$$

$$\mathrm{D}_{\langle a}S_{bc\rangle} = \mathrm{D}_{(a}S_{bc)} - \frac{2}{5}h_{(ab}(\operatorname{div} S)_{c)}\,. \tag{13}$$

We decompose the covariant derivatives of scalars, vectors, and rank-2 tensors into irreducible components

$$\nabla_a f = -\dot{f}u_a + \mathrm{D}_a f\,, \tag{14}$$

$$\nabla_b V_a = -\left(\dot{V}_{\langle a\rangle}u_b + u_a u_b \dot{u}_c V^c - \frac{1}{3}\Theta u_a V_b - u_a\sigma_{bc}V^c - u_a[\omega, V]_b\right) + \mathrm{D}_a V_b\,, \tag{15}$$

$$\begin{aligned}\nabla_c S_{ab} = &-\left(\dot{S}_{\langle ab\rangle}u_c + 2u_{(a}S_{b)d}\dot{u}^d u_c - \frac{2}{3}\Theta u_{(a}S_{b)c} - 2u_{(a}S_{b)}{}^d\sigma_{dc}\right.\\ &\left. -\ 2\varepsilon_{cde}u_{(a}S_{b)}{}^d\omega^e\right) + \mathrm{D}_a S_{bc}\,,\end{aligned} \tag{16}$$

where

$$\mathrm{D}_a V_b = \frac{1}{3}\mathrm{D}_c V^c h_{ab} - \frac{1}{2}\varepsilon_{abc}\operatorname{curl} V^c + \mathrm{D}_{\langle a}V_{b\rangle}\,, \tag{17}$$

$$\mathrm{D}_a S_{bc} = \frac{3}{5}\mathrm{D}^d S_{d\langle a}h_{b\rangle c} - \frac{2}{3}\varepsilon_{dc(a}\operatorname{curl} S_{b)}{}^d + \mathrm{D}_{\langle a}S_{bc\rangle}\,. \tag{18}$$

We also introduce the kinematic quantities encoding the relative motion of fluids:

$$\nabla_b u_a = \mathrm{D}_b u_a - \dot{u}_a u_b\,, \tag{19}$$

$$\mathrm{D}_b u_a = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}\,, \tag{20}$$

where $\dot{u}_a = u^b\nabla_b u_a$ is the relativistic acceleration vector, in the frames of instantaneously comoving observers $\dot{u}_a = \dot{u}_{\langle a\rangle}$, $\Theta = \mathrm{D}^a u_a$ the rate of expansion of fluids, $\sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle} = \mathrm{D}_{(a}u_{b)} - \frac{1}{3}h_{ab}\mathrm{D}_c u^c$, a traceless symmetric tensor ($\sigma_{ab} = \sigma_{(ab)}$, $\sigma_a{}^a = 0$); the shear tensor describing the rate of distortion of fluids, and $\omega_{ab} = \mathrm{D}_{[a}u_{b]}$ a skew-symmetric tensor ($\omega_{ab} = \omega_{[ab]}$, $\omega_a{}^a = 0$); the vorticity tensor describing the rotation of fluids.[27,33,37]

The vorticity vector[38,39] $\omega_a$ is defined by

$$\omega_a = -\frac{1}{2}\varepsilon_{abc}\omega^{bc}\,, \tag{21}$$

where $\omega_a u^a = 0$, $\omega_{ab}\omega^b = 0$ and the magnitude $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab} \geq 0$ have been imposed. Accordingly, we obtain

$$\omega_a = -\frac{1}{2}\varepsilon_{abc}\mathrm{D}^b u^c\,. \tag{22}$$

The sign convention is such that in the Newtonian theory $\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}$.

We denote the covariant shear and vorticity products of the symmetric traceless tensors as

$$[\sigma, S]_a = \varepsilon_{abc}\sigma^b{}_d S^{cd}\,, \qquad [\omega, S]_{\langle ab\rangle} = \varepsilon_{cd\langle a}S_{b\rangle}{}^c\omega^d\,. \tag{23}$$

The energy density and pressure of fluids are encoded in the dynamic quantities, which generally have the contributions from the energy flux and anisotropic pressure:

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}\,, \tag{24}$$

$$q_a u^a = 0\,, \qquad \pi^a{}_a = 0\,, \qquad \pi_{ab} = \pi_{(ab)}\,, \qquad \pi_{ab}u^b = 0\,, \tag{25}$$

where $\rho = T_{ab}u^a u^b$ is the relativistic energy density relative to $u^a$, $p = \frac{1}{3}T_{ab}h^{ab}$ the pressure, $q_a = -T_{\langle a\rangle b}u^b = -h_a{}^c T_{cb}u^b$ the energy flux relative to $u^a$, and $\pi_{ab} = T_{\langle ab\rangle} = T_{cd}h^c{}_{\langle a}u^d{}_{b\rangle} = (h^c{}_{(a}u^d{}_{b)} - \frac{1}{3}h_{ab}h^{cd})T_{cd}$ the traceless anisotropic stress. Imposing $q^a = \pi_{ab} = 0$, we get the solution of a perfect fluid with $T_{ab} = \rho u_a u_b + p h_{ab}$. In addition $p = 0$ gives the pressure-free matter or dust solution.[27,33,37]

## 3. Cosmological Field Equations

In the theory of general relativity, we describe the local nature of gravitational field nearby matter as an algebraic relation between the Ricci curvature and the matter fields, i.e. the Einstein field equations:

$$R_{ab} = T_{ab} - \frac{1}{2}T g_{ab}\,, \tag{26}$$

where $R_{ab}$ is the Ricci curvature, $T_{ab}$ the energy–momentum of the matter fields, and $T = T_c{}^c$ the trace of the energy–momentum tensor.

The successive contractions of Eq. (26) on using Eq. (24) lead to a set of relations:

$$\begin{gathered} R_{ab}u^a u^b = \frac{1}{2}(\rho + 3p)\,, \qquad h_a{}^b R_{bc}u^c = -q_a\,, \\ h_a{}^c h_b{}^d R_{cd} = \frac{1}{2}(\rho - p)h_{ab} + \pi_{ab}\,, \end{gathered} \tag{27}$$

$$R = R_a{}^a\,, \qquad T = T_a{}^a = -\rho + 3p\,, \qquad R = -T\,, \tag{28}$$

where $R$ is the Ricci scalar. The Ricci curvature is derived from the once contracted Riemann curvature tensor: $R_{ab} = R^c{}_{acb}$.

The Riemann tensor is split into symmetric (massless) traceless $C_{abcd}$ and traceful massive $M_{abcd}$ parts:

$$R_{abcd} = C_{abcd} + M_{abcd}\,. \tag{29}$$

The symmetric traceless part of the Riemann curvature is called the Weyl conformal curvature with the following properties:

$$C_{abcd} = C_{[ab][cd]}\,, \qquad C^a{}_{bca} = 0 = C_{a[bcd]}\,. \tag{30}$$

The nonlocal (long-range) fields, the parts of the curvature not directly determined locally by matter, are given by the Weyl curvature; propagating the Newtonian (and anti-Newtonian) forces and gravitational waves. It can be shown that the Weyl tensor $C_{abcd}$ is irreducibly split into the Newtonian $C^{\mathrm{N}}_{abcd}$ and the anti-Newtonian $C^{\mathrm{AN}}_{abcd}$ parts:

$$C_{abcd} = C^{\mathrm{N}}_{abcd} + C^{\mathrm{AN}}_{abcd}\,, \tag{31}$$

$$C^{ab}_{\mathrm{N}\;cd} = 4\{u^{[a}u_{[c} + h^{[a}{}_{[c}\}E^{b]}{}_{d]}\,, \tag{32}$$

$$C^{\mathrm{AN}}_{abcd} = 2\varepsilon_{abe}u_{[c}H_{d]}{}^{e} + 2\varepsilon_{cde}u_{[a}H_{b]}{}^{e}\,, \tag{33}$$

where $E_{ab} = C_{acbd}u^c u^d$ is the gravitoelectric field and $H_{ab} = \frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e$ the gravitomagnetic field. They are spacelike and traceless symmetric.

The traceful massive part of the Riemann curvature consists of the matter fields and the characteristics of local interactions with matter

$$\begin{aligned} M^{ab}{}_{cd} = {}& \frac{2}{3}(\rho + 3p)u^{[a}u_{[c}h^{b]}{}_{d]} + \frac{2}{3}\rho h^{a}{}_{[c}h^{b}{}_{d]} \\ & - 2u^{[a}h^{b]}{}_{[c}q_{d]} - 2u_{[c}h^{[a}{}_{d]}q^{b]} - 2u^{[a}u_{[c}\pi^{b]}{}_{d]} + 2h^{[a}{}_{[c}\pi^{b]}{}_{d]}\,. \end{aligned} \tag{34}$$

Therefore, the Weyl curvature is linked to the matter fields through the Riemann curvature.

### 3.1. *Dynamic formulas*

To provide equations governing relativistic dynamics of matter, we use the *Bianchi identities*

$$\nabla_{[e}R_{ab]cd} = 0\,. \tag{35}$$

On substituting Eq. (29) into Eq. (35), we get the dynamic formula for the Weyl conformal curvature[3,40,41]:

$$\nabla^d C_{abcd} = -\nabla_{[a}\left(R_{b]c} - \frac{1}{6}g_{b]c}R\right) = -\nabla_{[a}\left(T_{b]c} - \frac{1}{3}g_{b]c}T_d{}^d\right) \equiv J_{abc}\,. \tag{36}$$

On decomposing Eq. (36) along and orthogonal to a four-velocity vector, we obtain constraint ($C^{1,2}{}_a$) and propagation ($P^{1,2}{}_{ab}$) equations of the Weyl fields in a form analogous to the Maxwell equations[10,42–44]:

$$\begin{aligned} C^1{}_a \equiv {}& (\operatorname{div} E)_a - 3\omega^b H_{ab} - [\sigma, H]_a - \frac{1}{3}\mathrm{D}_a\rho + \frac{1}{3}\Theta q_a \\ & - \frac{1}{2}\sigma_{ab}q^b + \frac{3}{2}[\omega, q]_a + \frac{1}{2}(\operatorname{div}\pi)_a = 0\,, \end{aligned} \tag{37}$$

$$\begin{aligned} C^2{}_a \equiv {}& (\operatorname{div} H)_a + 3\omega^b E_{ab} + [\sigma, E]_a + \omega_a(\rho + p) \\ & + \frac{1}{2}\operatorname{curl}(q)_a + \frac{1}{2}[\sigma, \pi]_a - \frac{1}{2}\omega^b\pi_{ab} = 0\,, \end{aligned} \tag{38}$$

$$\begin{aligned} P^1{}_{ab} \equiv\ & \mathrm{curl}(H)_{ab} + 2[\dot{u}, H]_{\langle ab\rangle} - \dot{E}_{\langle ab\rangle} - \Theta E_{ab} + [\omega, E]_{\langle ab\rangle} \\ & + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \frac{1}{2}\sigma_{ab}(\rho + p) - \frac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} - \dot{u}_{\langle a}q_{b\rangle} \\ & - \frac{1}{2}\dot{\pi}_{\langle ab\rangle} - \frac{1}{6}\Theta\pi_{ab} + \frac{1}{2}[\omega, \pi]_{\langle ab\rangle} - \frac{1}{2}\sigma^e{}_{\langle a}\pi_{b\rangle e} = 0\,, \end{aligned} \tag{39}$$

$$\begin{aligned} P^2{}_{ab} \equiv\ & \mathrm{curl}(E)_{ab} + 2[\dot{u}, E]_{\langle ab\rangle} + \dot{H}_{\langle ab\rangle} + \Theta H_{ab} - [\omega, H]_{\langle ab\rangle} \\ & - 3\sigma_{c\langle a}H_{b\rangle}{}^c - \frac{3}{2}\omega_{\langle a}q_{b\rangle} - \frac{1}{2}[\sigma, q]_{\langle ab\rangle} - \frac{1}{2}\,\mathrm{curl}(\pi)_{ab} = 0\,. \end{aligned} \tag{40}$$

The twice contracted Bianchi identities present the conservation of the total energy–momentum tensor, namely

$$\nabla^b T_{ab} = \nabla^b\left(R_{ab} - \frac{1}{2}g_{ab}R\right) = 0\,. \tag{41}$$

It is split into a timelike and a spacelike momentum constraints:

$$C^3 \equiv \dot{\rho} + (\rho + p)\Theta + \mathrm{div}(q) + 2\dot{u}_a q^a + \sigma_{ab}\pi^{ab} = 0\,, \tag{42}$$

$$\begin{aligned} C^4{}_a \equiv\ & (\rho + p)\dot{u}_a + \mathrm{D}_a p + \dot{q}_{\langle a\rangle} + \frac{4}{3}\Theta q_a + \sigma_{ab}q^b \\ & - [\omega, q]_a + (\mathrm{div}\,\pi)_a + \dot{u}^b\pi_{ab} = 0\,. \end{aligned} \tag{43}$$

They provide the conservation law of energy–momentum, i.e. how matter determines the geometry, and describe the motion of matter.

### 3.2. *Kinematic formulas*

To provide the equations of motion, we use the *Ricci identities* for the vector field $u_a$:

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d\,. \tag{44}$$

We substitute the vector field $u_a$ from the kinematic quantities, using the Einstein equation, and separating out the orthogonally projected part into trace, symmetric traceless, and skew symmetric parts. We obtain constraints and propagations for the kinematic quantities as follows[42]:

$$P^3 \equiv \dot{\Theta} + \frac{1}{3}\Theta^2 - \mathrm{div}(\dot{u}) - \dot{u}^a\dot{u}_a - (\omega_{ab}\omega^{ab} - \sigma_{ab}\sigma^{ab}) + \frac{1}{2}(\rho + 3p) = 0\,, \tag{45}$$

$$P^4{}_a \equiv \dot{\omega}_{\langle a\rangle} + \frac{2}{3}\Theta\omega_a - \sigma_a{}^b\omega_b + \frac{1}{2}\,\mathrm{curl}(\dot{u})_a = 0\,, \tag{46}$$

$$\begin{aligned} P^5{}_{ab} \equiv\ & E_{ab} - \mathrm{D}_{\langle a}\dot{u}_{b\rangle} - \dot{u}_{\langle a}\dot{u}_{b\rangle} + \dot{\sigma}_{\langle ab\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \frac{2}{3}\sigma_{ab}\Theta + \omega_{\langle a}\omega_{b\rangle} \\ & - \frac{1}{2}\pi_{ab} = 0\,. \end{aligned} \tag{47}$$

Equation (45), called the Raychaudhuri propagation formula, is the basic equation of gravitational attraction.[45] In Eq. (46), the evolution of vorticity is conserved by the rotation of acceleration. Equation (47) shows that the gravitoelectric field is propagated in shear, vorticity, acceleration, and anisotropic stress.

The Ricci identities also provide a set of constraints:

$$C^5 \equiv \mathrm{div}(\omega) - \omega_a \dot{u}^a = 0\,, \tag{48}$$

$$C^6{}_a \equiv \frac{2}{3}\mathrm{D}_a\Theta - (\mathrm{div}\,\sigma)_a + \mathrm{curl}(\omega)_a + 2[\dot{u},\omega]_a - q_a = 0\,, \tag{49}$$

$$C^7{}_{ab} \equiv H_{ab} - \mathrm{curl}(\sigma)_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle} = 0\,. \tag{50}$$

Equation (48) presents the divergence of vorticity. Equation (49) links the divergence of shear to the rotation of vorticity. Equation (50) characterizes the gravitomagnetic field as the distortion of vorticity and the rotation of shear.

## 4. Gravitational Waves

We now obtain the temporal evolution of the dynamic propagations ($P^{1,2}{}_{ab}$) in a perfect-fluid model ($q^a = \pi_{ab} = 0$):

$$C^1{}_a = (\mathrm{div}\,E)_a - 3\omega^b H_{ab} - [\sigma,H]_a - \frac{1}{3}\mathrm{D}_a\rho = 0\,, \tag{51}$$

$$C^2{}_a = (\mathrm{div}\,H)_a + 3\omega^b E_{ab} + [\sigma,E]_a + \omega_a(\rho+p) = 0\,, \tag{52}$$

$$\begin{aligned} P^1{}_{ab} &= \mathrm{curl}(H)_{ab} + 2[\dot{u},H]_{\langle ab\rangle} - \dot{E}_{\langle a\rangle} - \Theta E_{ab} + [\omega,E]_{\langle ab\rangle} \\ &\quad + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \frac{1}{2}\sigma_{ab}(\rho+p) = 0\,, \end{aligned} \tag{53}$$

$$\begin{aligned} P^2{}_{ab} &= \mathrm{curl}(E)_{ab} + 2[\dot{u},E]_{\langle ab\rangle} + \dot{H}_{\langle ab\rangle} + \Theta H_{ab} - [\omega,H]_{\langle ab\rangle} \\ &\quad - 3\sigma_{c\langle a}H_{b\rangle}{}^c = 0\,. \end{aligned} \tag{54}$$

To first order, the evolution of propagation is

$$\begin{aligned} \dot{P}^1{}_{ab} &= \mathrm{D}^2 E_{ab} - \ddot{E}_{\langle ab\rangle} - \frac{3}{2}\mathrm{D}_{\langle a}C^1{}_{b\rangle} - \frac{4}{3}\Theta P^1{}_{ab} + \mathrm{curl}(P^2)_{ab} \\ &\quad - \frac{4}{3}\Theta^2 E_{ab} - \frac{7}{3}\Theta\dot{E}_{\langle ab\rangle} - \dot{\Theta}E_{ab} - \Theta E_{c\langle a}\sigma_{b\rangle}{}^c - \sigma_{cd}E^{cd}\sigma_{ab} \\ &\quad + E^{cd}\sigma_{ca}\sigma_{bd} - \sigma^{cd}\sigma_{c(a}E_{b)d} + \varepsilon_{cd(a}\dot{E}_{b)}{}^c\omega^d + \varepsilon_{cd(a}E_{b)}{}^c\dot{\omega}^d \\ &\quad + \frac{4}{3}\Theta[\omega,E]_{\langle ab\rangle} + 4\Theta\sigma_{c\langle a}E_{b\rangle}{}^c + \dot{\varepsilon}_{cd(a}E^c_{b)}\omega^d + 3\dot{\sigma}_{c\langle a}E_{b\rangle}{}^c \\ &\quad + 3\sigma_{c\langle a}\dot{E}_{b\rangle}{}^c - 2\,\mathrm{curl}([\dot{u},E])_{ab} - \frac{1}{2}\mathrm{D}_{\langle a}\omega^c H_{b\rangle c} \\ &\quad - \frac{3}{2}\mathrm{D}_{\langle a}[\sigma,H]_{b\rangle} + \frac{8}{3}\Theta[\dot{u},H]_{\langle ab\rangle} + \mathrm{curl}([\omega,H])_{ab} \end{aligned}$$

$$+ 3\,\mathrm{curl}(\sigma_{c\langle a}H_{b\rangle}{}^c) - \sigma_e{}^c\varepsilon_{cd(a}\mathrm{D}^eH_{b)}{}^d$$
$$+ 2\varepsilon_{cd(a}\dot{H}_{b)}{}^c\dot{u}^d + 2\varepsilon_{cd(a}H_{b)}{}^c\ddot{u}^d + 2\dot{\varepsilon}_{cd(a}H_{b)}{}^c\dot{u}^d - \frac{1}{2}\sigma_{ab}(\dot{\rho}+\dot{p})$$
$$- \frac{1}{3}\Theta\sigma_{ab}(\rho+p) = 0\,. \tag{55}$$

We neglect products of kinematic quantities with respect to the undisturbed metrics (unexpansive static spacetime). We can also prevent the perturbations that are merely associated with coordinate transformation, since they have no physical significance. In free space, we get

$$\dot{P}^1{}_{ab} = \mathrm{D}^2E_{ab} - \ddot{E}_{\langle ab\rangle} - \frac{3}{2}\mathrm{D}_{\langle a}C^1{}_{b\rangle} - \frac{4}{3}\Theta P^1{}_{ab} + \mathrm{curl}(P^2)_{ab} = 0\,. \tag{56}$$

To be consistent with Eqs. (51)–(54), $\mathrm{D}^2E_{ab} - \ddot{E}_{\langle ab\rangle}$ has to vanish. Similarly, the evolution of $P^2{}_{ab}$ shows that $\mathrm{D}^2H_{ab} - \ddot{H}_{\langle ab\rangle} = 0$. The evolutions reflect that the divergenceless and nonvanishing rotation of the Weyl fields are necessary conditions for gravitational waves:

$$(\mathrm{div}\,E)_a = (\mathrm{div}\,H)_a = 0\,, \qquad \mathrm{curl}(E)_{ab} \neq 0 \neq \mathrm{curl}(H)_{ab}\,. \tag{57}$$

Indeed, the rotation of the Weyl fields characterizes the wave solutions. The gravitomagnetic field is explicitly important to describe the gravitational waves, and is comparable with the Maxwell fields.

We use Eq. (18) to provide two more constraints:

$$C^8{}_{abc} \equiv \mathrm{D}_aE_{bc} - \mathrm{D}_{\langle a}E_{bc\rangle} - \frac{3}{5}\mathrm{D}^dE_{d\langle a}h_{b\rangle c} + \frac{2}{3}\varepsilon_{dc(a}\,\mathrm{curl}(E)_{b)}{}^d = 0\,, \tag{58}$$

$$C^9{}_{abc} \equiv \mathrm{D}_aH_{bc} - \mathrm{D}_{\langle a}H_{bc\rangle} - \frac{3}{5}\mathrm{D}^dH_{d\langle a}h_{b\rangle c} + \frac{2}{3}\varepsilon_{dc(a}\,\mathrm{curl}(H)_{b)}{}^d = 0\,. \tag{59}$$

To first order, divergence of Eq. (58) is

$$\mathrm{D}^aC^8{}_{abc} = \mathrm{D}^2E_{bc} - \mathrm{D}^a\mathrm{D}_{\langle a}E_{bc\rangle} - \frac{3}{5}\mathrm{D}^a\mathrm{D}^dE_{d\langle a}h_{b\rangle c}$$
$$+ \frac{1}{3}\mathrm{D}^a\varepsilon_{dca}\,\mathrm{curl}(E)_b{}^d + \frac{1}{3}\varepsilon_{dcb}\,\mathrm{D}^a\,\mathrm{curl}(E)_a{}^d = 0\,. \tag{60}$$

On substituting Eq. (54), it becomes

$$\mathrm{D}^aC^8{}_{abc} = \mathrm{D}^2E_{bc} - \mathrm{D}^a\mathrm{D}_{\langle a}E_{bc\rangle} - \frac{3}{5}\mathrm{D}^a\mathrm{D}^dE_{d\langle a}h_{b\rangle c} - \frac{3}{5}\mathrm{D}^aC^1{}_{\langle a}h_{b\rangle c}$$
$$+ \frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}P^2{}_{b)d} - \frac{9}{5}\mathrm{D}^a\omega^dH_{d\langle a}h_{b\rangle c} - \frac{3}{5}\mathrm{D}^a[\sigma,H]_{\langle a}h_{b\rangle c}$$
$$- \frac{1}{5}\mathrm{D}^a\mathrm{D}_{\langle a}\rho h_{b\rangle c} - \frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}\dot{H}_{\langle b)d\rangle} - \frac{4}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}[\dot{u},E]_{\langle b)d\rangle}$$
$$- \Theta\frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}H_{b)d} + \frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}[\omega,H]_{\langle b)d\rangle}$$
$$+ 2\mathrm{D}^a\varepsilon^d{}_{c(a}\sigma_{c\langle b)}H_{d\rangle}{}^c = 0\,. \tag{61}$$

We abandon products of kinematic quantities in the undisturbed metrics:

$$\mathrm{D}^aC^8{}_{abc} = \mathrm{D}^2E_{bc} - \mathrm{D}^a\mathrm{D}_{\langle a}E_{bc\rangle} - \frac{3}{5}\mathrm{D}^a\mathrm{D}^dE_{d\langle a}h_{b\rangle c} - \frac{3}{5}\mathrm{D}^aC^1{}_{\langle a}h_{b\rangle c} + \frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}P^2{}_{b)d} - \frac{2}{3}\mathrm{curl}(\dot{H})_{ab} = 0\,. \quad (62)$$

To linearized order, we get $\mathrm{curl}(S_{ab})^{\cdot} = \mathrm{curl}\,\dot{S}_{ab}$. Using the later point and the evolution of Eq. (53), we obtain:

$$\mathrm{D}^aC^8{}_{abc} = \mathrm{D}^2E_{bc} - \mathrm{D}^a\mathrm{D}_{\langle a}E_{bc\rangle} - \frac{3}{5}\mathrm{D}^a\mathrm{D}^dE_{d\langle a}h_{b\rangle c} - \frac{2}{3}\ddot{E}_{\langle ab\rangle} - \frac{3}{5}\mathrm{D}_{\langle a}C^1{}_{b\rangle} + \frac{2}{3}\mathrm{D}^a\varepsilon^d{}_{c(a}P^2{}_{b)d} - \frac{2}{3}\dot{P}^1{}_{ab} = 0\,. \quad (63)$$

The result can be compared to the wave solution (56). Without the distortion parts, it is inconsistent with a generic description of wave. Distortion of the gravitoelectric field ($\mathrm{D}_{\langle a}E_{bc\rangle}$) must not vanish to provide the wave solution. We also obtain similar condition for the gravitomagnetic field. In free space, the divergence of the Weyl fields, determined by the matter, must be free. The temporal evolution decides that the rotation of the Weyl fields must be nonzero. Now, the distortion provides another condition to characterize the evolution of the Weyl fields:

$$\mathrm{D}_{\langle a}E_{bc\rangle} \neq 0 \neq \mathrm{D}_{\langle a}H_{bc\rangle}\,. \quad (64)$$

The existence of rotation and distortion is necessary condition to maintain the wave solutions.

## 5. Newtonian and Anti-Newtonian Fields

We can associate a Newtonian model with purely gravitoelectric ($H_{ab} = 0$). Without the gravitomagnetism, the nonlocal nature of the Newtonian force cannot be retrieved from relativistic models. It also excludes gravitational waves. The Newtonian model is a limited model to show the characteristics of the gravitoelectric. We can also consider an anti-Newtonian model; a model with purely gravitomagnetic ($E_{ab} = 0$). The anti-Newtonian model obstructs sounding solutions. In Ref. 43, it has been proven that the anti-Newtonian model shall include either shear or vorticity.

### 5.1. *Newtonian model*

Let us consider the Newtonian model ($H_{ab} = 0$) in an irrotational static spacetime ($\omega_a = \dot{u}_a = 0$) and a perfect-fluid model ($q^a = \pi_{ab} = 0$). The constraints and propagations shall be

$$C^1{}_a = (\mathrm{div}\,E)_a - \frac{1}{3}\mathrm{D}_a\rho = 0\,, \qquad C^2{}_a = [\sigma, E]_a = 0\,, \quad (65)$$

$$P^1{}_{ab} = -\dot{E}_{\langle ab\rangle} - \Theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \frac{1}{2}\sigma_{ab}(\rho + p) = 0\,, \qquad P^2{}_{ab} = \mathrm{curl}(E)_{ab} = 0\,, \tag{66}$$

$$C^6{}_a = \frac{2}{3}\mathrm{D}_a\Theta - (\mathrm{div}\,\sigma)_a = 0\,, \qquad C^7{}_{ab} = -\mathrm{curl}(\sigma)_{ab} = 0\,. \tag{67}$$

To first order, divergence and evolution of Eq. (66b) are

$$\begin{aligned} \mathrm{D}^b P^2{}_{ab} &= \frac{1}{2}\varepsilon_{abc}\mathrm{D}^b(\mathrm{D}_d E^{cd}) + \frac{1}{3}\Theta[\sigma, E]_a - \sigma_{ab}[\sigma, E]^b \\ &= \frac{1}{2}\,\mathrm{curl}(C^1)_a + \frac{1}{3}\Theta C^2{}_a - \sigma_a{}^b C^2{}_b + \frac{1}{3}\omega_a\dot{\rho}\,, \end{aligned} \tag{68}$$

$$\begin{aligned} \dot{P}^2{}_{ab} &= -\frac{1}{3}\Theta\,\mathrm{curl}(E)_{ab} - \sigma_e{}^c\varepsilon_{cd(a}\mathrm{D}^e E_{b)}{}^d + \mathrm{curl}(\dot{E})_{ab} \\ &= -\frac{3}{2}\varepsilon^{cd}{}_{(a}\sigma_{b)c}C^1{}_d - \frac{4}{3}\Theta P^2{}_{ab} + \frac{3}{2}\varepsilon^c{}_{d(a}C^6{}_c E_{b)}{}^d \\ &\quad - \frac{1}{2}(\rho + p)C^7{}_{ab} - \mathrm{curl}(P^1)_{ab} + 3\,\mathrm{curl}(\sigma_{c\langle a}E_{b\rangle}{}^c)\,. \end{aligned} \tag{69}$$

The last parameter ($\frac{1}{3}\omega_a\dot{\rho}$) in Eq. (68) vanishes because of irrotational condition. Equation (68) then conserves the constraints. Equation (69) must be consistent with Eqs. (65) and (67). Thus, the last term in Eq. (69) has to vanish:

$$\mathrm{curl}(\sigma_{c\langle a}E_{b\rangle}{}^c) = 0\,. \tag{70}$$

It is a necessary condition for the consistent evolution of propagation. This condition is satisfied with irrotational product of gravitoelectric and shear, but it is a complete contrast to Eq. (65b). Thus, the Newtonian model is generally inconsistent with generic relativistic models. Moreover, the temporal evolution of propagation shows no wave solutions.

#### 5.1.1. *Newtonian limit*

The Newtonian model obstructs wave solution, due to the instantaneous interaction. Following Refs. 28–30, we consider a model whose action propagates at infinite speed ($c \to \infty$). This is compatible with $\lim_{c\to\infty} E_{ab} = E_{ab}(t)|_\infty$, where $E_{ab}(t)|_\infty$ is an arbitrary function of time.

We define the Newtonian potential as

$$E_{ab} \equiv \mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\Phi = \mathrm{D}_a\mathrm{D}_b\Phi - \frac{1}{3}h_{ab}\mathrm{D}^2\Phi\,. \tag{71}$$

On substituting into Eq. (65a), we get

$$C^1{}_a = \mathrm{D}_a\mathrm{D}^2\Phi - \frac{1}{3}\mathrm{D}^b h_{ab}\mathrm{D}^2\Phi - \frac{1}{3}\mathrm{D}_a\rho = 0\,. \tag{72}$$

In a spatial infinity, we obtain the Poisson equation of the Newtonian potential:

$$C^1 \equiv \mathrm{D}^2\Phi - \frac{1}{2}\rho = 0\,. \tag{73}$$

Equation (65a) generalizes the gravitoelectric as the Newtonian force in the gradient of the relativistic energy density.

Moreover, Eq. (66a) gives

$$\begin{aligned} {P^1}_{ab} = -\mathrm{D}_a\mathrm{D}_b\dot{\Phi} + \frac{1}{3}h_{ab}\mathrm{D}^2\dot{\Phi} - \Theta\mathrm{D}_a\mathrm{D}_b\Phi + \frac{1}{3}(\dot{h}_{ab} + \Theta h_{ab})\mathrm{D}^2\Phi \\ + 3\sigma_{c\langle a}\mathrm{D}_{b\rangle}\mathrm{D}^c\Phi - \sigma_{c\langle a}{h_{b\rangle}}^c\mathrm{D}^2\Phi - \frac{1}{2}\sigma_{ab}(\rho + p) = 0\,. \end{aligned} \tag{74}$$

In the Newtonian theory, we could not find the temporal evolution of the Newtonian potential.

#### 5.1.2. *Acceleration potential*

In an irrotational spacetime, Eq. (46) becomes

$${P^4}_a = \frac{1}{2}\,\mathrm{curl}(\dot{u})_a = 0\,. \tag{75}$$

It introduces a scalar potential:

$$\dot{u}_a = \mathrm{D}_a\Phi\,, \tag{76}$$

where $\Phi$ is the acceleration potential. This scalar potential corresponds to the Newtonian potential. In the irrotational Newtonian model, the linearized acceleration is characterized as the acceleration potential.

### 5.2. *Anti-Newtonian model*

Let us consider the anti-Newtonian model ($E_{ab} = 0$) in a shearless static spacetime ($\omega_a = \dot{u}_a = 0$) and a perfect-fluid model ($q^a = \pi_{ab} = 0$). The constraints and propagations shall be

$${C^1}_a = -3\omega^b H_{ab} - \frac{1}{3}\mathrm{D}_a\rho = 0\,, \qquad {C^2}_a = (\mathrm{div}\,H)_a + \omega_a(\rho + p) = 0\,, \tag{77}$$

$${P^1}_{ab} = \mathrm{curl}(H)_{ab} = 0\,, \qquad {P^2}_{ab} = \dot{H}_{\langle ab\rangle} + \Theta H_{ab} - [\omega, H]_{\langle ab\rangle} = 0\,, \tag{78}$$

$${C^6}_a = \frac{2}{3}\mathrm{D}_a\Theta + \mathrm{curl}(\omega)_a = 0\,, \qquad {C^7}_{ab} = H_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle} = 0\,. \tag{79}$$

To linearized order, divergence and evolution of Eq. (78a) are

$$\begin{aligned} \mathrm{D}^b{P^1}_{ab} &= \frac{1}{2}\varepsilon_{abc}\mathrm{D}^b(\mathrm{D}_d H^{cd}) \\ &= \frac{1}{2}{\varepsilon_{ab}}^c\mathrm{D}^b{C^2}_c - \frac{1}{2}(\rho + p){C^6}_a + \frac{1}{3}(\rho + p)\mathrm{D}_a\Theta\,, \end{aligned} \tag{80}$$

$$\begin{aligned}\dot{P}^1{}_{ab} &= -\frac{1}{3}\Theta\,\mathrm{curl}(H)_{ab} + \mathrm{curl}(\dot{H})_{ab}\\ &= -\frac{4}{3}\Theta P^1{}_{ab} + \mathrm{curl}(P^2)_{ab} + \mathrm{curl}([\omega, H])_{\langle ab\rangle}\,. \end{aligned}\tag{81}$$

Equation (80) is consistent only in the spacetime being free from either the gravitational mass and pressure or the gradient of expansion. According to Eqs. (77) and (79), the last term in Eq. (81) has to vanish:

$$\mathrm{curl}([\omega, H])_{\langle ab\rangle} = 0\,. \tag{82}$$

It is a necessary condition for the consistent evolution of propagation. This condition is satisfied with irrotational vorticity products of gravitomagnetic, but it is not consistent with Eq. (79b):

$$\begin{aligned}&\varepsilon^c{}_{d(a}C^7{}_{b)c}\omega^d - \frac{1}{4}\omega_b C^6{}_a - \frac{1}{4}\omega_a C^6{}_b - \frac{1}{4}\mathrm{D}_b[\omega,\omega]_a - \frac{1}{4}\mathrm{D}_a[\omega,\omega]_b\\ &\quad + \frac{1}{6}\omega_b \mathrm{D}_a\Theta + \frac{1}{6}\omega_a\mathrm{D}_b\Theta - \varepsilon^c{}_{da}\dot{u}_{\langle b}\omega_{c\rangle}\omega^d - \varepsilon^c{}_{db}\dot{u}_{\langle a}\omega_{c\rangle}\omega^d = 0\,. \end{aligned}\tag{83}$$

Thus, the anti-Newtonian model is generally inconsistent with relativistic models. Furthermore, there is no possibility of gravitational waves.

#### 5.2.1. *Vorticity potential*

In an unexpansive spacetime, Eq. (79a) takes the following form:

$$C^6{}_a = \mathrm{curl}(\omega)_a = 0\,. \tag{84}$$

It defines a vorticity scalar potential $\Psi$ as

$$\omega_a = \mathrm{D}_a\Psi\,. \tag{85}$$

In the unexpansive anti-Newtonian model, the linearized vorticity is characterized as the vorticity potential.

#### 5.2.2. *Anti-Newtonian limit*

We may consider a gravitomagnetic model whose action propagates at infinite speed. Let us define the anti-Newtonian potential as

$$H_{ab} \equiv \mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\Psi = \mathrm{D}_a\mathrm{D}_b\Psi - \frac{1}{3}h_{ab}\mathrm{D}^2\Psi\,. \tag{86}$$

We substitute Eqs. (85) and (86) into Eq. (77b):

$$C^2{}_a = \mathrm{D}_a\mathrm{D}^2\Psi - \frac{1}{3}\mathrm{D}^b h_{ab}\mathrm{D}^2\Psi + \mathrm{D}_a\Psi(\rho + p) = 0\,. \tag{87}$$

In a spatial infinity, we derive the Helmholtz equation:

$$C^2 \equiv \mathrm{D}^2\Psi + \frac{3}{2}(\rho + p)\Psi = 0\,. \tag{88}$$

Equation (77b) associates the gravitomagnetic with the angular momentum $\omega_a(\rho + p)$.

## 6. Conclusion

The Weyl curvature tensor describes the nonlocal long-range interactions as enabling gravitational act at a distance (tidal forces and gravitational waves). The gravitoelectric field is described as the relativistic generalization of the tidal (Newtonian) force. However, the gravitomagnetic (anti-Newtonian) force has no Newtonian analogue. We have no expression similar to $\dot{E}_{ab}$ in the Newtonian theory. This difference arises from the instantaneous action in the Newtonian theory, which excludes a sounding solution. In Sec. 4, the rotation and distortion of the Weyl fields characterize the gravitational wave. The gravitomagnetism is necessary to maintain the gravitational wave. In relativistic models, the Newtonian force is also inconsistent without the magnetic part of the Weyl curvature.

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