

## Gravitational fields of the magnetic-type\*

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Local conformal symmetry introduces the conformal curvature (Weyl tensor) that gets split into its (gravito-) electric and magnetic (tensor) parts. Newtonian tidal forces are expected from the gravitoelectric field, whereas general-relativistic frame-dragging effects emerge from the gravitomagnetic field. The symmetric, traceless gravitoelectric and gravitomagnetic tensor fields can be visualized by their eigenvectors and eigenvalues. In this paper, we depict the gravitoelectric and gravitomagnetic fields around a slowly rotating black hole. This suggests that the phenomenon of ultra-fast outflows observed at the centers of active galaxies may give evidence for the gravitomagnetic fields of spinning supermassive black holes. We also question whether the current issues in our contemporary observations might be resolved by the inclusion of gravitomagnetism on large scales in a perturbed FLRW model.

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The applicability and validity of Newtonian gravity and classical cosmology have been challenged in both the weak-gravity limit on large scales and the strong-gravity regime near supermassive black holes (SMBH). In particular, our observations of Type Ia supernovae up to the redshift  $z \sim 2$  suggested the accelerating expansion of the universe,<sup>1</sup> which was interpreted as dark energy.<sup>2</sup> Meanwhile, the rotational velocity curves of visible stars in *disc* (spiral) galaxies are inconsistent with Kepler's laws of planetary motion,<sup>3</sup> which were explained by cold dark matter halos enveloping galactic discs.<sup>4</sup> Moreover, our contemporary high-energy observations suggested the presence of *ultra-fast outflows* with nearly relativistic velocities originated from somewhere close to SMBHs in *active* galaxies and quasars.<sup>5</sup> Recently, our understanding of the universe has been revolutionized by the discovery of gravitational

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waves resulting from a merger of binary stellar-mass black holes<sup>6</sup> and binary neutron stars,<sup>7</sup> which were predicted by the theory of general relativity in 1916.<sup>8</sup> In general relativity, we also had the prediction of a non-Newtonian field that is called the *gravitomagnetic* field<sup>9</sup> by analogy with the magnetic field in Maxwell’s theory of electromagnetism. The Lense–Thirring frame-dragging effect<sup>10</sup> that is one of the footprints of *gravitomagnetism* has been recently detected in a fast-rotating white dwarf in a binary system.<sup>11</sup> This effect was previously measured around the Earth using two artificial satellites.<sup>12</sup>

Conformal invariance of Maxwell’s equations in electromagnetism has inspired us to explore conformal transformations in other fundamental forces of the nature. Considering a local conformal (Weyl) transformation of the metric,  $g_{ab} \rightarrow \Omega^2 g_{ab}$  (where  $\Omega^2$  is the position-dependent conformal factor), we had the introduction of the *Weyl conformal tensor*  $C_{abcd}$  to the Riemann curvature  $R_{abcd}$ .<sup>13</sup> The Weyl tensor  $C_{abcd}$  is conformally invariant and has only 10 independent components. A conformal theory of gravity (conformal Weyl gravity) was prescribed by an action given by the square of the Weyl tensor<sup>14</sup> that seems to be spontaneously broken (similar to the BEH mechanism) in some energy scales, leading to the Einstein–Hilbert action for the Einstein field equations.<sup>15</sup> The Weyl tensor can be split into its electric and magnetic parts, i.e. the *gravitoelectric tensor* field  $E_{ab} \equiv c^2 C_{acbd}(u^c/c)(u^d/c)$  and the *gravitomagnetic tensor* field  $H_{ab} \equiv -\frac{1}{2}\epsilon_{aecdc} C^{cd}{}_{bf}(u^e/c)(u^f/c)$ , where  $u^a/c$  is the normalized timelike vector field (such that  $u^a u_a = -c^2$ ),  $\epsilon_{abcd}$  is the space-time permutation tensor and  $c$  is the speed of light in vacuum. The gravitoelectric and gravitomagnetic fields are the spatial symmetric, traceless tensors ( $E_{ab} = E_{ba}$ ,  $H_{ab} = H_{ba}$  and  $E_a{}^a = 0 = H_a{}^a$ ), and each has five independent components. Equations of motion for the gravitoelectric and gravitomagnetic fields are obtained by substituting the Einstein field equations into the Bianchi identities.<sup>16</sup> Let us consider a perfect fluid model with  $T_{ab} = (\rho c^2 + p)(u_a/c)(u_b/c) + p\eta_{ab}$  that is commonly employed in almost-FLRW spacetimes, where  $\eta_{ab} = \text{diag}(-c^2, +1, +1, +1)$  is the Minkowski metric,  $\rho c^2$  is the energy density ( $\rho$  is the volumetric mass density) and  $p$  is the isotropic pressure. For a nonexpanding nonaccelerated shearless model in a locally almost flat coordinate system, these equations of motion for  $E_{ab}$  and  $H_{ab}$  become

$$D^b E_{ab} - 3c\omega^b H_{ab} = \frac{8\pi G}{3} D_a \rho, \tag{1}$$

$$D^b H_{ab} + \frac{3}{c^3} \omega^b E_{ab} = -\frac{8\pi G}{c^3} \omega_a (\rho + p/c^2), \tag{2}$$

$$\text{curl}(E)_{ab} = -c \frac{dH_{ab}}{dt} - c H_{c(a} \omega_{b)}{}^c, \tag{3}$$

$$\text{curl}(H)_{ab} = \frac{1}{c^3} \frac{dE_{ab}}{dt} + \frac{1}{c^3} E_{c(a} \omega_{b)}{}^c, \tag{4}$$

where  $G$  is the gravitational constant,  $t$  is the time coordinate,  $D_a \equiv \partial/\partial x^a$  denotes the spatial derivative with respect to the space coordinates  $x^a$ ,

$\text{curl}(S)_{ab} \equiv \epsilon_{cd(a} D^c S_b)^d$  denotes the spatial curl of second-rank spatial symmetric tensors,  $\omega_{ab} \equiv D_{[a} u_{b]}$  is the vorticity tensor, and  $u_b$  is the velocity vector,  $\omega_a \equiv -\frac{1}{2} \epsilon_{abc} D^b u^c$  is the vorticity vector, and  $\epsilon_{abc} \equiv \epsilon_{abcd}(u^d/c)$  is the spatial permutation tensor. The round brackets enclosing indices denote symmetrization (e.g.  $A_{(ab)} \equiv \frac{1}{2} A_{ab} + \frac{1}{2} A_{ba}$ ), whereas the square brackets enclosing indices denote antisymmetrization (e.g.  $A_{[ab]} \equiv \frac{1}{2} A_{ab} - \frac{1}{2} A_{ba}$ ).

Newtonian tidal forces are produced by the gravitoelectric field  $E_{ab}$ , while frame-dragging effects are generated by the gravitomagnetic field  $H_{ab}$ . In the first two Eqs. (1) and (2), the spatial gradient of the mass density,  $(8\pi G/3)D_a \rho$ , and the angular momentum density,  $-(8\pi G/c^3)\omega_a(\rho + p/c^2)$ , appear as matter sources for the gravitoelectric and gravitomagnetic fields, respectively. In the Newtonian limit ( $D^b E_{ab} = (8\pi G/3)D_a \rho$  and  $H_{ab} = 0$ ), taking  $E_{ab} = D_a D_b \Phi - \frac{1}{3} h_{ab} D^2 \Phi$  leads to Poisson's equation of Newtonian gravity  $D^2 \Phi = 4\pi G \rho$ , where  $D^2 \equiv D_a D^a$  is the Laplace operator and  $h_{ab} = \text{diag}(+1, +1, +1)$  is the spatial flat metric. The later two Eqs. (3) and (4) support the wave solutions for the gravitoelectric and gravitomagnetic fields, i.e.  $D^2 E_{ab} - (1/c^2)d^2 E_{ab}/dt^2 = 0$  and  $D^2 H_{ab} - (1/c^2)d^2 H_{ab}/dt^2 = 0$  (see also Ref. 17) in vacuum where the vorticity and matter fields vanish.

We note an invariance in Eqs. (1)–(4) between the gravitoelectric and gravitomagnetic tensors,  $(E_{ab}/c^2, H_{ab}) \rightarrow (-H_{ab}, E_{ab}/c^2)$ , as well as the mass density spatial gradient and the angular momentum density,  $(\frac{1}{3}D_a \rho, -\omega_a(\rho + p/c^2)/c) \rightarrow (\omega_a(\rho + p/c^2)/c, \frac{1}{3}D_a \rho)$ , which demonstrate a type of the SO(2) electric–magnetic duality.<sup>18</sup> The gravitoelectric and gravitomagnetic fields are transformed into each other under the electric–magnetic duality rotations, which are analogous with the electric–magnetic invariance in Maxwell's equations. However, the angular momentum density appears as a source for the gravitomagnetic field  $H_{ab}$  in general relativity, whereas we have no magnetic charge for the magnetic field  $\mathbf{H}$  in Maxwell's theory of electromagnetism.

The gravitoelectric tensor fields  $E_{ab}$  generated around a massive object with the mass quantity are comparable to the electric vector fields  $\mathbf{E}$  around a charged particle with the charge quantity. The gravitomagnetic tensor fields  $H_{ab}$  produced around a rotating massive object having the angular momentum may be compared with the magnetic vector fields  $\mathbf{H}$  around a bar magnet having the magnetic dipole moment. Nevertheless, we have the second-rank symmetric traceless tensor fields in gravity rather than the vector (first-rank tensor) fields in electromagnetism. We can visualize the physical lines and amplitudes of the tensor fields by obtaining their eigenvectors and eigenvalues.<sup>19</sup> Accordingly, the physical properties of  $E_{ab}$  and  $H_{ab}$  have been visualized based on integral curves of their eigenvectors, the so-called *tendex* and *vortex* lines, respectively.<sup>20,21</sup>

In general relativity, we describe a black hole (BH) by three fundamental quantities: mass  $M$ , spin  $a_*$  and charge  $Q$ .<sup>22</sup> The dimensionless spin parameter ( $-1 \leq a_* \leq +1$ ) is defined as  $a_* = Jc/GM^2$ , where  $J$  is the BH angular momentum and  $M$  is the BH mass. Negative values of  $a_*$  describe retrograde rotation

in which the black hole rotates in the opposite direction to the accretion disk, while positive values are associated with prograde rotation, and  $a_* = 0$  implies no rotation. As a charged black hole would be rapidly neutralized by the accretion of oppositely charged particles, the charge quantity  $Q$  could be negligible. Let us consider a slow rotating BH described by Ref. 21 in the Boyer–Lindquist coordinates (radial distance:  $r$ , polar angle:  $\theta$ , azimuthal angle:  $\varphi$ ). In the Kerr metric, we use the Kerr spin parameter  $a \equiv a_* GM/c^2$  with the dimension of length. We then obtain the gravitoelectric and gravitomagnetic tensor fields for a slow rotating SMBH with a spin parameter of  $a_* = 0.5$  and a mass of  $M = 10^8 M_\odot$ , and solve  $E_{ab}V_\mathcal{E}^a = \lambda_\mathcal{E}V_b^\mathcal{E}$  and  $H_{ab}V_\mathcal{H}^a = \lambda_\mathcal{H}V_b^\mathcal{H}$  for the radial distance  $r$ , where  $V_a^\mathcal{E}$  and  $V_a^\mathcal{H}$  are the eigenvectors, and  $\lambda^\mathcal{E}$  and  $\lambda^\mathcal{H}$  are the eigenvalues of  $E_{ab}$  and  $H_{ab}$ , respectively. The absolute values of the radial distance eigenvalues  $|\lambda_r^\mathcal{E}|$  and  $|\lambda_r^\mathcal{H}|$  correspond to the amplitudes of the gravitoelectric tensor  $E_{ab}$  and gravitomagnetic tensor  $H_{ab}$  as functions of the radial distance  $r$ , respectively. The radial distance eigenvectors  $\mathbf{V}_r^\mathcal{E}$  and  $\mathbf{V}_r^\mathcal{H}$  visualize the physical lines of the gravitoelectric and gravitomagnetic tensor fields. Figure 1 shows  $\log_{10} |\lambda_r^\mathcal{E}|$  and  $\log_{10} c^2 |\lambda_r^\mathcal{H}|$  by color codes, and  $\text{sgn}(\lambda_r^\mathcal{E})\mathbf{V}_r^\mathcal{E}$  and  $\text{sgn}(\lambda_r^\mathcal{H})\mathbf{V}_r^\mathcal{H}$  by vector arrows ( $\text{sgn}(x)$  is the signum function). The physical lines shown for  $\text{sgn}(\lambda_r^\mathcal{H})\mathbf{V}_r^\mathcal{H}$  are comparable to the unified outflow model proposed for ultra-fast outflows observed in high-energy X-ray observations of active galactic nuclei.<sup>23</sup> In particular, the measurements of SMBH spins are now possible with the recent advancements in X-ray astronomy,<sup>24,25</sup> so we could examine whether SMBH angular momenta are correlated with outflow kinematics and density profiles. From Fig. 1, it can be seen that  $|\lambda_r^\mathcal{H}|$  has its maximum value at regions at the north and south poles outside the event horizon, while it vanishes at the boundary of the event horizon, so the gravitomagnetic field may support outflows of accreted materials along the BH spin axis far from the event horizon of the spinning BH. This is in agreement with the Penrose mechanism<sup>26</sup> that explained how rotational energy is extracted from a Kerr BH.

We might generalize the *slow-Kerr* metric of Ref. 21 to a perturbed FLRW spacetime that is applicable to a large supermassive noncompact object slowly rotating in almost-FLRW model such as a massive disc galaxy. To describe a galaxy, the BH spin parameter  $a_*$  is replaced with a dimensionless spin parameter  $\lambda_*$ , so we may define a perturbed FLRW spacetime as follows:

$$\begin{aligned}
 ds^2 = & - \left( 1 + \frac{2\Phi(M, r, \theta)}{c^2} \right) c^2 dt^2 + \left( 1 + \frac{2\Phi(M, r, \theta)}{c^2} \right)^{-1} dr^2 \\
 & + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{4\lambda\Phi(M, r, \theta)}{c^2} \sin^2 \theta c dt d\varphi,
 \end{aligned} \tag{5}$$

where  $\Phi(M, r, \theta)$  is the Newtonian gravitational potential, and  $\lambda \equiv \lambda_* GM/c^2$  is a parameter with the dimension of length that characterizes the rotation.

For disc-like galaxies, we may define the gravitational potential  $\Phi$  based on the generalized Pulmmer’s<sup>27</sup> three-dimensional mass model in the spherical coordinates

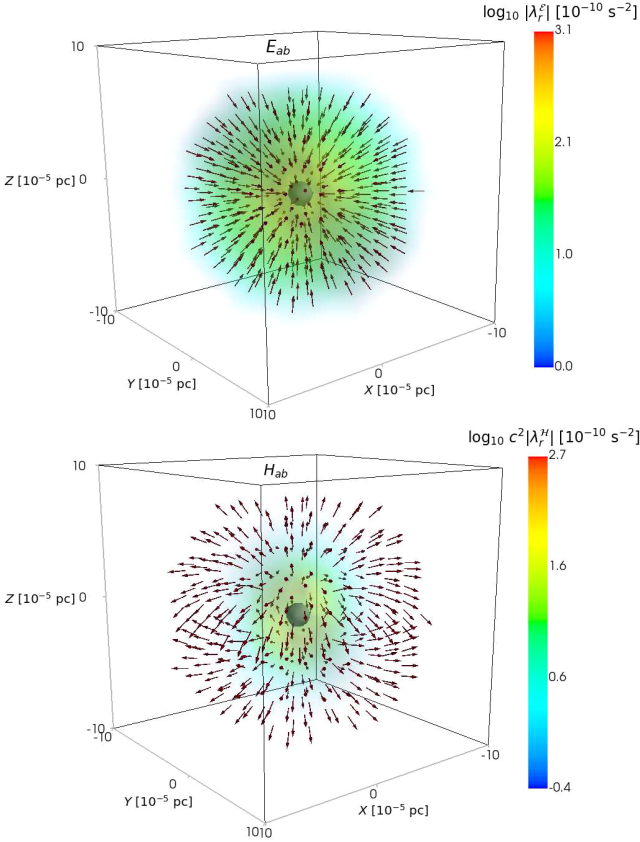


Fig. 1. (Color online) The gravitoelectric tensor field  $E_{ab}$  (top panel) and gravitomagnetic tensor field  $H_{ab}$  (bottom panel) of a slowly rotating SMBH with the dimensionless spin parameter of  $a_* = 0.5$  and mass of  $M = 10^8 M_\odot$ . The color codes show the absolute value of the radial distance eigenvalues  $\log_{10} |\lambda_r^E|$  and  $\log_{10} c^2 |\lambda_r^H|$ , the vector arrows visualize the radial distance eigenvectors regulated by the signs of the radial distance eigenvalues,  $\text{sgn}(\lambda_r^E) \mathbf{V}_r^E$  and  $\text{sgn}(\lambda_r^H) \mathbf{V}_r^H$ .

as follows<sup>28</sup>:

$$\Phi(M, r, \theta) = -\frac{GM}{(r^2 + [R_a + (R_b^2 + r^2 \cos^2 \theta)^{1/2}]^2)^{1/2}}, \quad (6)$$

and the dimensionless spin parameter  $\lambda_*$  as<sup>29</sup>

$$\lambda_* = \frac{J|E|^{1/2}}{GM^{5/2}}, \quad (7)$$

where  $R_a$  and  $R_b$  are constants with the dimension of length characterizing various nonspheroidal mass distributions,  $J$  is the total angular momentum,  $E$  is the total binding energy and  $M$  is the total mass. The spin parameter is typically a low value around  $\lambda_* \approx 0.05$  for elliptical (*nondisc*) galaxies, but a larger value about  $\lambda_* \approx 0.5$  reported for spiral and lenticular (*disc*) galaxies.<sup>30</sup>

In this configuration, the weak production of gravitomagnetic fields on both sides of the galactic disc might be expected by the rotation of a massive spiral galaxy typically having baryonic masses of  $10^{8.5} - 10^{11.5} M_{\odot}$ <sup>31</sup> and spins of  $\lambda_* \approx 0.5$ .<sup>30</sup> In the case of an *active* galaxy containing a rapidly spinning SMBH at its center, we may also expect the strong production of gravitomagnetic fields near the galactic center along the spin axis powered by the spinning SMBH typically having masses of  $10^6 - 10^9 M_{\odot}$ <sup>32</sup> and some having spins of  $a_* \approx 0.9$  (measured in several active galaxies from relativistically broadened X-ray  $K\alpha$  iron lines<sup>25</sup>). This phenomenon can be explained by Eq. (2) that associates the gravitomagnetic field production with the angular momentum density. Other possible phenomena are predicted by Eqs. (3) and (4) where the curl (and temporal variation) of the gravitomagnetic field contributes to the temporal variation (and curl) of the gravitoelectric field.

Can the rotation of a massive disc galaxy and its rapidly spinning SMBH contribute to the production of gravitomagnetic fields on both sides of the galactic disc and along the SMBH spin axis? Can these gravitomagnetic fields cycling over the galactic disc induce some gravitoelectric fields into rotational motions of stars within the galactic disc? Implications of the conformal Weyl gravity for galactic rotation curves have been explored by Ref. 14, but using the Schwarzschild solution, which could not adequately explain discrepancies in rotational velocity curves between elliptical galaxies ( $\lambda_* \approx 0.05$ ) and spiral (disc) galaxies ( $\lambda_* \approx 0.5$ ). It is worthwhile considering whether the current issues in rotation curves of disc galaxies could be resolved by the equations of motion for the gravitoelectric and gravitomagnetic fields in a perturbed FLRW model.

How could be the interaction between two massive *active* galaxies due to their weak gravitomagnetic fields on large scales? In particular, some recent  $N$ -body computational simulations of a perturbed FLRW spacetime<sup>33</sup> imply that the frame-dragging vortex, which is expected to be large on small scales (e.g. near SMBH), could be at smaller orders but considerable on large scales, and be also enhanced as the universe is evolving from the primeval at the redshift  $z \sim 10$  to the present-day one at  $z \sim 0$ . We know that the universe just after cosmic reionization ( $z \sim 6$ ) contained mostly low-mass *starburst* dwarf galaxies, which were gradually evolving into massive *quiescent* and *active* galaxies at cosmic noon ( $z \sim 1.5-3$ ) due to multiple galaxy merger events. We do not yet fully comprehend how this galaxy evolution influenced our universe on large scales.

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